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An Example of a Goldie Ring Whose Matrix Ring Is Not Goldie

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Goldie's theorem characterizes semiprime Goldie rings as orders in semisimple Artinian rings. If a ring is an order in an Artinian ring then all its matrix rings are also orders in Artinian rings [1]. A long-standing question has been whether the matrix rings over an arbitrary Goldie ring are again Goldie rings. By the example presented here, that question is now resolved in the negative.

In this paper we construct a commutative Goldie ring whose 2×2 matrix ring has an infinite ascending chain of annihilators. Thus this matrix ring, and so the ring itself, cannot be embedded in an Artinian ring. Therefore this example also resolves the question whether commutative Goldie rings can be embedded in Artinian rings.

A ring is said to be a Goldie ring if it has the ascending chain condition (ACC) on annihilator right ideals, and if it has finite rank, i.e., the maximum condition on direct sums of right ideals.

It is well known that a matrix ring over a ring with finite rank also has finite rank. Thus, if the matrix ring over a Goldie ring fails the Goldie criteria, it must have an infinite ascending chain of annihilators.

LEMMA. *Let* $\mathbf{R}$ *be an arbitrary ring. Suppose* $\mathbf{R}$ *contains two infinite sets of elements,* $\{\mathbf{s}_i\}$, $\{\mathbf{t}_i\}$, $i = 1, 2, 3,\ldots$, *such that* $\mathbf{s}_i\mathbf{t}_j = 0$ *iff* $i \neq j$. *Then* $\mathbf{R}$ *has an infinite ascending chain of annihilators.*

Proof. Consider $B_k = \{\mathbf{s}_i \mid i \geqslant k\}$. Since $B_k \supset B_{k+1}$, $\operatorname{Ann} B_k \subset \operatorname{Ann} B_{k+1}$. Let $A_k = \operatorname{Ann} B_k$. Now $\mathbf{s}_k\mathbf{t}_k \neq 0$ implies $\mathbf{t}_k \notin A_k$. But $\mathbf{t}_k \in A_{k+1}$ since $\mathbf{s}_l\mathbf{t}_k = 0$ for all $l \neq k$. Therefore we have $A_1 \not\subset A_2 \not\subset A_3 \not\subset \cdots$, an infinite ascending chain of annihilators in $\mathbf{R}$.

(In fact, in an arbitrary ring R, there exists an infinite ascending chain of annihilators iff there exist two infinite sets $\{s_i\}$, $\{t_i\}$ in R such that $s_it_j = 0$ for $i > j$, $s_it_i \neq 0$.)

We will apply the lemma in the following way. Suppose we have a ring R with elements $\{a_i, b_i, c_i, d_i\}$, $i = 1, 2, 3, \ldots$. Consider

$$s_i = \begin{bmatrix} a_i & -b_i \\ 0 & 0 \end{bmatrix}, \qquad t_j = \begin{bmatrix} c_j & 0 \\ d_j & 0 \end{bmatrix}$$

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in the matrix ring R_2. Then

$$s_i t_j = \begin{bmatrix} a_i c_i - b_i d_j & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $\gamma_{ij} = a_i c_j - b_i d_j$. Suppose there exists an ideal I in R such that $\gamma_{ij} \in I$ iff $i \neq j$. Then in the ring $\bar{R} = R/I$, we have $\bar{\gamma}_{ij} = 0$ iff $i \neq j$ or equivalently, $\bar{s}_i \bar{t}_j = 0$ in $\bar{R}_2$ iff $i \neq j$. Thus by the above lemma, $\bar{R}_2$ has an infinite ascending chain of annihilators.

Now we proceed to define a ring R with such an ideal I and such that $\bar{R} = R/I$ is a Goldie ring. We choose I so that it is "nicely" contained in a prime ideal P. This will greatly facilitate proving $\bar{R}$ to be Goldie.

The following notation will be used. Let K be an arbitrary commutative integral domain, $A = \{a_i, b_i, c_i, d_i\}$, $X = \{x_i, y_i, u\}$, $i = 1, 2, 3,\ldots$, two sets of indeterminates over K, $\gamma_{ij} = a_i c_j - b_i d_j$, $R = K[A]$, $D = K[X]$. We make R and D into graded rings by giving elements of A, x_i, y_i degree 1 and elements of K and u degree 0.

To obtain a prime ideal P containing γ_{ij} we consider the graded K-algebra homomorphism $f: R \to D$ such that $f: a_i \to u x_i$, $b_i \to x_i$, $c_j \to y_j$, $d_j \to u y_j$. Note that $f(\gamma_{ij}) = f(a_i c_j - b_i d_j) = 0$. Let $P = \ker f$ which is prime since D is a domain, and which is homogeneous since f is a graded homomorphism. So $P = \sum P_i$, where $P_i = \{p \in P$ such that p is homogeneous of degree i in $A\}$. Note that since f maps monomials to monomials, $P_0 = P_1 = (0)$ and P_2 is the free K-module with basis $B = \{\gamma_{ij}, a_i b_j - a_j b_i, c_i d_j - c_j d_i\}$.

Since $\gamma_{ii} \in P$ we need to find a slightly smaller ideal I such that γ_{ii} is not in I, but γ_{ij} is in I for $i \neq j$. Let I be the ideal generated by $\bigcup_{h=3}^{\infty} P_h$, $\{g \in B \mid g \neq \gamma_{ii}$, for any $i\}$ and $\{\gamma_{ii} - \gamma_{jj}$, for all $i, j\}$. Let $w = \bar{\gamma}_{ii} \neq 0$ in $\bar{R}$, $\bar{R} = R/I$. Note that for any element $r \notin I$, $r \in P$ iff $r = \sum k_i \gamma_{ii}$, $k_i \in K$. In particular, for $\bar{r} \neq 0$, $\bar{r} \in \bar{P}$ iff $\bar{r} = kw$, $k \in K$, $k \neq 0$. So $\bar{P}$ is isomorphic to K as a K-module and hence has rank 1. Note also that $\bar{P}$ is prime. Let $\bar{Q}$ be the prime ideal of $\bar{R}$ generated by the image of A. Note that $\bar{Q} = \operatorname{Ann} \bar{P}$.

We have now defined the ideal I such that $\gamma_{ij} \in I$ iff $i \neq j$, so we need only show $\bar{R}$ to be Goldie. We begin by showing that there are only two nontrivial annihilator ideals, namely, $\bar{P}$ and $\bar{Q}$.

Take a nonempty subset S of $\bar{R}$ such that $S \neq \{0\} \neq \operatorname{Ann} S$. Note that if $S \subset \bar{P}$ then $\operatorname{Ann} S = \bar{Q}$. If $S \not\subset \bar{P}$ then $\operatorname{Ann} S \subset \bar{P}$ since $\bar{P}$ is prime. Thus $\operatorname{Ann}(\operatorname{Ann} S) = \bar{Q}$ and $\operatorname{Ann} S = \operatorname{Ann} \bar{Q} = \bar{P}$. So, with only finitely many annihilator ideals, $\bar{R}$ surely satisfies the ACC on annihilators.

Next we compute the rank of $\bar{R}$. Suppose J_1, J_2 are ideals in $\bar{R}$ such that $J_1 \cap J_2 = 0$, $J_1 \neq 0 \neq J_2$. Then from $J_1 J_2 = 0$, J_1 or $J_2 \subset \bar{P}$, but not both since $\bar{P}$ has rank 1. Without loss of generality let $J_1 \subset \bar{P}$, $J_2 \not\subset \bar{P} = 0$. Let J' be any other ideal in $\bar{R}$ such that $J' \cap J_1 \oplus J_2 = 0$. Then $J' J_2 = 0$ so $J' \subset \bar{P}$. Thus J', $J_1 \subset \bar{P}$ and $J' \cap J_1 = 0$. Since $\bar{P}$ has rank 1, $J' = 0$. Therefore $\bar{R}$ has rank 2.

It is of interest to note that all polynomial rings over $\bar{R}$ remain Goldie rings. If T is any new set of indeterminates over $\bar{R}$ apply the above construction starting with the domain $K[T]$. The resulting Goldie ring is isomorphic to $\bar{R}[T]$. So even if every polynomial ring over a Goldie ring is a Goldie ring, the matrix rings need not be Goldie rings. Just what additional conditions on a Goldie ring will ensure the Goldie conditions for its matrix rings are unknown.

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Reference

1. L. W. Small, Orders in Artinian rings, *J. Algebra* 4 (1966), 13–41.